

# GENERALIZATIONS OF ANDREWS' CURIOUS IDENTITIES

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ABSTRACT. According to the method of series rearrangement, we establish two generalizations of Andrews' curious  $q$ -series identity with an extra integer parameter. The limiting cases of them produce two extensions of Andrews' curious  ${}_3F_2(\frac{3}{4})$ -series identity with an additional integer parameter. Meanwhile, several related results are also given.

## 1. INTRODUCTION

For a complex variable  $x$ , define the shifted-factorial by

$$(x)_n = \begin{cases} \prod_{k=0}^{n-1} (x+k), & n > 0; \\ 1, & n = 0; \\ \frac{(-1)^n}{\prod_{k=1}^n (k-x)}, & n < 0. \end{cases}$$

For simplifying the expressions, we shall use the symbol:

$$\left[ \begin{matrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{matrix} \right]_n = \frac{(a)_n (b)_n \cdots (c)_n}{(\alpha)_n (\beta)_n \cdots (\gamma)_n}.$$

Following Bailey [3], define the hypergeometric series by

$${}_{{1+r}}F_s \left[ \begin{matrix} a_0, & a_1, & \cdots, & a_r \\ b_1, & \cdots, & b_s \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_r)_k}{k! (b_1)_k \cdots (b_s)_k} z^k,$$

where  $\{a_i\}_{i \geq 0}$  and  $\{b_j\}_{j \geq 1}$  are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then the curious  ${}_3F_2(\frac{3}{4})$ -series identity due to Andrews [1, Equation (22b)] reads as

$${}_3F_2 \left[ -n, \quad \frac{a}{2}, \quad \frac{3a+n}{2} \middle| \frac{3}{4} \right] = \left[ \frac{\frac{1}{3}}{\frac{1}{3}+a}, \quad \frac{\frac{2}{3}}{\frac{2}{3}+a} \right]_{\frac{n}{3}} \chi(n), \quad (1)$$

where the symbol  $\chi(n)$  has been offered by

$$\chi(n) = \begin{cases} 1, & n = 3m; \\ 0, & \text{otherwise.} \end{cases}$$

The reversal of it is the  ${}_3F_2(\frac{4}{3})$ -series identity:

$${}_3F_2 \left[ -n, \quad \frac{3b}{2}, \quad \frac{1+3b}{2} \middle| \frac{4}{3} \right] = \left[ \frac{\frac{1}{3}}{\frac{1}{3}-b}, \quad \frac{\frac{2}{3}}{\frac{2}{3}+b} \right]_{\frac{n}{3}} \chi(n). \quad (2)$$

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Some recent research work for hypergeometric series should be mentioned. In accordance with inversion techniques, Chen and Chu [4, 5] derived many  ${}_3F_2(\frac{4}{3})$ -series identities related to (2). According to series rearrangement, Chu [6], Lavoie [7] and Lavoie et al. [8] explored  ${}_3F_2(1)$ -series identities.

For two complex numbers  $x$  and  $q$ , define the  $q$ -shifted factorial by

$$(x; q)_n = \begin{cases} \prod_{i=0}^{n-1} (1 - xq^i), & n > 0; \\ 1, & n = 0; \\ \frac{1}{\prod_{j=n}^{-1} (1 - xq^j)}, & n < 0. \end{cases}$$

The fraction form of it reads as

$$\left[ \begin{matrix} a, & b, & \cdots, & c \\ \alpha, & \beta, & \cdots, & \gamma \end{matrix} \middle| q \right]_n = \frac{(a; q)_n (b; q)_n \cdots (c; q)_n}{(\alpha; q)_n (\beta; q)_n \cdots (\gamma; q)_n}.$$

Following Gasper and Rahman [9], define the  $q$ -series by

$${}_{1+r}\phi_s \left[ \begin{matrix} a_0, & a_1, & \cdots, & a_r \\ b_1, & \cdots, & b_s \end{matrix} \middle| q; z \right] = \sum_{k=0}^{\infty} \left[ \begin{matrix} a_0, & a_1, & \cdots, & a_r \\ q, & b_1, & \cdots, & b_s \end{matrix} \middle| q \right]_k z^k,$$

where  $\{a_i\}_{i \geq 0}$  and  $\{b_j\}_{j \geq 1}$  are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then the terminating  ${}_6\phi_5$ -series identity (cf. Gasper and Rahman [9, p. 42]) can be expressed as

$${}_6\phi_5 \left[ \begin{matrix} a, & q\sqrt{a}, & -q\sqrt{a}, & b, & c, & q^{-\ell} \\ \sqrt{a}, & -\sqrt{a}, & qa/b, & qa/c, & q^{1+\ell}a \end{matrix} \middle| q; \frac{q^{1+\ell}a}{bc} \right] = \left[ \begin{matrix} qa, & qa/bc \\ qa/b, & qa/c \end{matrix} \middle| q \right]_{\ell}. \quad (3)$$

Recall a curious  $q$ -series identity due to Andrews [2, Equation (4.7)]:

$$\sum_{k=0}^n (a; q^3)_k \left[ \begin{matrix} q^{-n}, & q^n a \\ q, & \sqrt{a}, & -\sqrt{a}, & \sqrt{qa}, & -\sqrt{qa} \end{matrix} \middle| q \right]_k q^k = a^{n/3} \left[ \begin{matrix} q, & q^2 \\ qa, & q^2 a \end{matrix} \middle| q^3 \right]_{\frac{n}{3}} \chi(n). \quad (4)$$

The reversal of it can be stated as

$$\sum_{k=0}^n \frac{1}{(q^{2-n}b; q^3)_k} \left[ \begin{matrix} q^{-n}, & \sqrt{b}, & -\sqrt{b}, & \sqrt{qb}, & -\sqrt{qb} \\ q, & b \end{matrix} \middle| q \right]_k q^k = \left[ \begin{matrix} q, & q^2 \\ q/b, & q^2 b \end{matrix} \middle| q^3 \right]_{\frac{n}{3}} \chi(n). \quad (5)$$

Inspired by the work of [4]-[8], we shall establish two generalizations of (4), which involve two extensions of (1), by means of series rearrangement in section 2. The reversal of them creates two generalizations of (5), which involve two extensions of (2), in section 3.

## 2. GENERALIZATIONS OF ANDREWS' CURIOUS IDENTITIES

**Theorem 1.** *For a nonnegative integer  $\ell$  and a complex number  $a$ , there holds*

$$\begin{aligned} & \sum_{k=0}^n (a; q^3)_k \left[ \begin{matrix} q^{-n}, & q^{n-\ell} a \\ q, & \sqrt{a}, & -\sqrt{a}, & \sqrt{qa}, & -\sqrt{qa} \end{matrix} \middle| q \right]_k q^k = \left[ \begin{matrix} q^{n-\ell} a \\ q^{2n-\ell} a \end{matrix} \middle| q \right]_{\ell} \\ & \times \sum_{i=0}^{\ell} a^{\frac{n+2i}{3}} q^{(\ell+2n-i)i} \frac{a - q^{2i-2n}}{a - q^{-2n}} \left[ \begin{matrix} q^{-\ell}, & q^{-n}, & q^{-2n}/a \\ q, & q^{1+\ell-2n}/a, & q^{n-i} a \end{matrix} \middle| q \right]_i \\ & \times \left[ \begin{matrix} q, & q^2 \\ qa, & q^2 a \end{matrix} \middle| q^3 \right]_{\frac{n-i}{3}} \chi(n-i). \end{aligned}$$

*Proof.* Letting  $a \rightarrow q^{-2n}/a$ ,  $b \rightarrow q^{k-n}$  and  $c \rightarrow 0$  for (3), we get the equation:

$$\left[ \begin{matrix} q^{n-\ell+k} a \\ q^{2n-\ell} a \end{matrix} \middle| q \right]_{\ell} \sum_{i=0}^{\ell} a^i q^{(\ell+2n-i)i} \frac{a - q^{2i-2n}}{a - q^{-2n}} \left[ \begin{matrix} q^{-\ell}, & q^{k-n}, & q^{-2n}/a \\ q, & q^{1+\ell-2n}/a, & q^{n+k-i} a \end{matrix} \middle| q \right]_i = 1.$$

Then there is the following relation:

$$\begin{aligned}
& \sum_{k=0}^n (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{qa}, -\sqrt{qa} \mid q \right]_k q^k \\
&= \sum_{k=0}^n (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{qa}, -\sqrt{qa} \mid q \right]_k q^k \\
&\times \left[ \frac{q^{n-\ell+k} a}{q^{2n-\ell} a} \mid q \right]_\ell \sum_{i=0}^{\ell} a^i q^{(\ell+2n-i)i} \frac{a - q^{2i-2n}}{a - q^{-2n}} \left[ \frac{q^{-\ell}, q^{k-n}, q^{-2n}/a}{q, q^{1+\ell-2n}/a, q^{n+k-i} a} \mid q \right]_i.
\end{aligned}$$

Interchange the summation order for the last double sum to achieve

$$\begin{aligned}
& \sum_{k=0}^n (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{qa}, -\sqrt{qa} \mid q \right]_k q^k \\
&= \left[ \frac{q^{n-\ell} a}{q^{2n-\ell} a} \mid q \right]_\ell \sum_{i=0}^{\ell} a^i q^{(\ell+2n-i)i} \frac{a - q^{2i-2n}}{a - q^{-2n}} \left[ \frac{q^{-\ell}, q^{-n}, q^{-2n}/a}{q, q^{1+\ell-2n}/a, q^{n-i} a} \mid q \right]_i \\
&\times \sum_{k=0}^{n-i} (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{qa}, -\sqrt{qa} \mid q \right]_k q^k.
\end{aligned}$$

Calculating the series on the last line by (4), we attain Theorem 1 to complete the proof.  $\square$

When  $\ell = 0$ , Theorem 1 reduces to (4) exactly. Other two examples are displayed as follows.

**Example 1** ( $\ell = 1$  in Theorem 1).

$$\begin{aligned}
& \sum_{k=0}^n (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{qa}, -\sqrt{qa} \mid q \right]_k q^k \\
&= \begin{cases} \frac{a^m(1-aq^{-1})}{1-aq^{6m-1}} \left[ \frac{q, q^2}{qa, q^{-1}a} \mid q^3 \right]_m, & n = 3m; \\ \frac{a^{m+1}q^{3m}(1-q)}{1-aq^{6m+1}} \left[ \frac{q^2, q^4}{qa, q^2a} \mid q^3 \right]_m, & n = 1 + 3m; \\ 0, & n = 2 + 3m. \end{cases}
\end{aligned}$$

**Example 2** ( $\ell = 2$  in Theorem 1).

$$\begin{aligned}
& \sum_{k=0}^n (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{qa}, -\sqrt{qa} \mid q \right]_k q^k \\
&= \begin{cases} \frac{a^m(1-aq^{-2})(1-aq^{-1})}{(1-aq^{6m-2})(1-aq^{6m-1})} \left[ \frac{q, q^2}{q^{-1}a, q^{-2}a} \mid q^3 \right]_m, & n = 3m; \\ \frac{a^{m+1}q^{3m-1}(1-q^2)(1-aq^{-1})}{(1-aq^{6m-1})(1-aq^{6m+1})} \left[ \frac{q^2, q^4}{q^{-1}a, qa} \mid q^3 \right]_m, & n = 1 + 3m; \\ \frac{a^{m+2}q^{6m}(1-q)(1-q^2)}{(1-aq^{6m+1})(1-aq^{6m+2})} \left[ \frac{q^4, q^5}{qa, q^2a} \mid q^3 \right]_m, & n = 2 + 3m. \end{cases}
\end{aligned}$$

Performing the replacement  $a \rightarrow q^{3a}$  in Theorem 1 and then letting  $q \rightarrow 1$ , we obtain the following equation.

**Theorem 2.** For a nonnegative integer  $\ell$  and a complex number  $a$ , there holds

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} -n, & a, & 3a+n-\ell \\ \frac{3a}{2}, & \frac{1+3a}{2} \end{matrix} \middle| \frac{3}{4} \right] = \left[ \begin{matrix} 1-n-3a \\ 1-2n-3a \end{matrix} \right]_{\ell} \\ & \times \sum_{i=0}^{\ell} (-1)^i \frac{3a+2n-2i}{3a+2n} \left[ \begin{matrix} -\ell, -n, -2n-3a \\ 1, 1-n-3a, 1+\ell-2n-3a \end{matrix} \right]_i \\ & \times \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ \frac{1}{3}+a, & \frac{2}{3}+a \end{matrix} \right]_{\frac{n-i}{3}} \chi(n-i). \end{aligned}$$

When  $\ell = 0$ , Theorem 2 reduces to (1) exactly. Other two examples are laid out as follows.

**Example 3** ( $\ell = 1$  in Theorem 2).

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} -n, & a, & 3a+n-1 \\ \frac{3a}{2}, & \frac{1+3a}{2} \end{matrix} \middle| \frac{3}{4} \right] \\ & = \begin{cases} \frac{3a-1}{3a+6m-1} \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ -\frac{1}{3}+a, & \frac{1}{3}+a \end{matrix} \right]_m, & n=3m; \\ \frac{1}{3a+6m+1} \left[ \begin{matrix} \frac{2}{3}, & \frac{4}{3} \\ \frac{1}{3}+a, & \frac{2}{3}+a \end{matrix} \right]_m, & n=1+3m; \\ 0, & n=2+3m. \end{cases} \end{aligned}$$

**Example 4** ( $\ell = 2$  in Theorem 2).

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} -n, & a, & 3a+n-2 \\ \frac{3a}{2}, & \frac{1+3a}{2} \end{matrix} \middle| \frac{3}{4} \right] \\ & = \begin{cases} \frac{(3a-2)(3a-1)}{(3a+6m-2)(3a+6m-1)} \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ -\frac{1}{3}+a, & -\frac{2}{3}+a \end{matrix} \right]_m, & n=3m; \\ \frac{2(3a-1)}{(3a+6m-1)(3a+6m+1)} \left[ \begin{matrix} \frac{2}{3}, & \frac{4}{3} \\ -\frac{1}{3}+a, & \frac{1}{3}+a \end{matrix} \right]_m, & n=1+3m; \\ \frac{2}{(3a+6m+1)(3a+6m+2)} \left[ \begin{matrix} \frac{4}{3}, & \frac{5}{3} \\ \frac{1}{3}+a, & \frac{2}{3}+a \end{matrix} \right]_m, & n=2+3m. \end{cases} \end{aligned}$$

**Theorem 3.** For a nonnegative integer  $\ell$  and a complex number  $a$ , there holds

$$\begin{aligned} & \sum_{k=0}^n (a; q^3)_k \left[ \begin{matrix} q^{-n}, q^{\frac{n-\ell}{2}} a \\ q, \sqrt{a}, -\sqrt{a}, \sqrt{q^{1-2\ell} a}, -\sqrt{qa} \end{matrix} \middle| q \right]_k q^k = \left[ \begin{matrix} q^{n-\ell} a, q^{\frac{1}{2}+n-\ell} \sqrt{a} \\ q^{2n-\ell} a, q^{\frac{1}{2}-\ell} \sqrt{a} \end{matrix} \middle| q \right]_{\ell} \\ & \times \sum_{i=0}^{\ell} a^{\frac{2n-5i}{6}} q^{(\frac{5}{2}+\ell)i} \frac{1-q^{2n-2i} a}{1-q^{2n} a} \left[ \begin{matrix} q^{-\ell}, q^{-n}, q^{-2n}/a \\ q, q^{1+\ell-2n}/a, q^{1-n}/a \end{matrix} \middle| q \right]_i \\ & \times \left[ \begin{matrix} q, q^2 \\ qa, q^2 a \end{matrix} \middle| q^3 \right]_{\frac{n-i}{3}} \chi(n-i). \end{aligned}$$

*Proof.* Letting  $a \rightarrow q^{-2n}/a$ ,  $b \rightarrow q^{k-n}$  and  $c \rightarrow q^{\frac{1}{2}-n}/\sqrt{a}$  for (3), the resulting equation reads as

$$\begin{aligned} & \left[ \begin{matrix} q^{n-\ell} a, q^{\frac{1}{2}+n-\ell} \sqrt{a} \\ q^{2n-\ell} a, q^{\frac{1}{2}+k-\ell} \sqrt{a} \end{matrix} \middle| q \right]_{\ell} \sum_{i=0}^{\ell} \frac{q^{(5/2+\ell)i}}{a^{i/2}} \frac{1-q^{2n-2i} a}{1-q^{2n} a} \left[ \begin{matrix} q^{-\ell}, q^{k-n}, q^{-2n}/a \\ q, q^{1+\ell-2n}/a, q^{1-n}/a \end{matrix} \middle| q \right]_i \\ & \times \left[ \begin{matrix} q^{n-\ell+k} a \\ q^{n-\ell} a \end{matrix} \middle| q \right]_{\ell-i} = 1. \end{aligned}$$

Then we can proceed as follows:

$$\begin{aligned}
& \sum_{k=0}^n (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{q^{1-2\ell}a}, -\sqrt{qa} \mid q \right]_k q^k \\
&= \sum_{k=0}^n (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{q^{1-2\ell}a}, -\sqrt{qa} \mid q \right]_k q^k \\
&\times \left[ q^{n-\ell}a, q^{\frac{1}{2}+n-\ell}\sqrt{a} \mid q \right] \sum_{\ell=0}^{\ell} \frac{q^{(5/2+\ell)i}}{a^{i/2}} \frac{1-q^{2n-2i}a}{1-q^{2n}a} \left[ q^{-\ell}, q^{k-n}, q^{-2n}/a \mid q \right]_i \\
&\times \left[ q^{n-\ell+k}a, q^{n-\ell}a \mid q \right]_{\ell-i}.
\end{aligned}$$

Interchange the summation order for the last double sum to get

$$\begin{aligned}
& \sum_{k=0}^n (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{q^{1-2\ell}a}, -\sqrt{qa} \mid q \right]_k q^k = \left[ q^{n-\ell}a, q^{\frac{1}{2}+n-\ell}\sqrt{a} \mid q \right]_{\ell} \\
&\times \sum_{i=0}^{\ell} \frac{q^{(5/2+\ell)i}}{a^{i/2}} \frac{1-q^{2n-2i}a}{1-q^{2n}a} \left[ q^{-\ell}, q^{-n}, q^{-2n}/a \mid q \right]_i \\
&\times \sum_{k=0}^{n-i} (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{qa}, -\sqrt{qa} \mid q \right]_k q^k.
\end{aligned}$$

Evaluating the series on the last line by (4), we achieve Theorem 3 to complete the proof.  $\square$

When  $\ell = 0$ , Theorem 3 also reduces to (4) exactly. Other two examples are displayed as follows.

**Example 5** ( $\ell = 1$  in Theorem 3).

$$\begin{aligned}
& \sum_{k=0}^n (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{q^{-1}a}, -\sqrt{qa} \mid q \right]_k q^k \\
&= \begin{cases} \frac{a^m(\sqrt{q}+\sqrt{a})}{\sqrt{q}+q^{3m}\sqrt{a}} \left[ q, q^2 \mid q^3 \right]_m, & n = 3m; \\ \frac{a^{m+1/2}(q-1)}{(\sqrt{q}-\sqrt{a})(1+q^{3m+1/2}\sqrt{a})} \left[ q^2, q^4 \mid q^3 \right]_m, & n = 1 + 3m; \\ 0, & n = 2 + 3m. \end{cases}
\end{aligned}$$

**Example 6** ( $\ell = 2$  in Theorem 3).

$$\begin{aligned}
& \sum_{k=0}^n (a; q^3)_k \left[ q, \sqrt{a}, -\sqrt{a}, \sqrt{q^{-3}a}, -\sqrt{qa} \mid q \right]_k q^k \\
&= \begin{cases} \frac{a^m(q^2-a)(\sqrt{q}+\sqrt{a})(1-q^{3m-3/2}\sqrt{a})}{(q^2-q^{6m}a)(1-q^{-3/2}\sqrt{a})(\sqrt{q}+q^{3m}\sqrt{a})} \left[ q, q^2 \mid q^3 \right]_m, & n = 3m; \\ \frac{a^{m+1/2}(q^2-1)(\sqrt{q}+\sqrt{a})}{(q^2-\sqrt{qa})(1+q^{3m-1/2}\sqrt{a})(1+q^{3m+1/2}\sqrt{a})} \left[ q^2, q^4 \mid q^3 \right]_m, & n = 1 + 3m; \\ \frac{a^{m+1}(1-q)(1-q^2)(1-q^{3m+3/2}\sqrt{a})}{(\sqrt{q}-\sqrt{a})(\sqrt{q^3}-\sqrt{a})(1+q^{3m+1/2}\sqrt{a})(1-q^{6m+2}a)} \left[ q^4, q^5 \mid q^3 \right]_m, & n = 2 + 3m. \end{cases}
\end{aligned}$$

Employing the substitution  $a \rightarrow q^{3a}$  in Theorem 3 and then letting  $q \rightarrow 1$ , we obtain the following equation.

**Theorem 4.** For a nonnegative integer  $\ell$  and a complex number  $a$ , there holds

$$\begin{aligned} {}_3F_2 \left[ \begin{matrix} -n, & a, & 3a+n-\ell \\ \frac{3a}{2}, & \frac{1+3a}{2}-\ell \end{matrix} \middle| \frac{3}{4} \right] &= \left[ \begin{matrix} 1-3a-n, & \frac{1-3a}{2}-n \\ 1-3a-2n, & \frac{1-3a}{2} \end{matrix} \right]_{\ell} \\ &\times \sum_{i=0}^{\ell} \frac{3a+2n-2i}{3a+2n} \left[ \begin{matrix} -\ell, -n, -2n-3a \\ 1, 1-n-3a, 1+\ell-2n-3a \end{matrix} \right]_i \\ &\times \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ \frac{1}{3}+a, & \frac{2}{3}+a \end{matrix} \right]_{\frac{n-i}{3}} \chi(n-i). \end{aligned}$$

When  $\ell = 0$ , Theorem 4 also reduces to (1) exactly. Other two examples are laid out as follows.

**Example 7** ( $\ell = 1$  in Theorem 4).

$$\begin{aligned} {}_3F_2 \left[ \begin{matrix} -n, & a, & 3a+n-1 \\ \frac{3a-1}{2}, & \frac{3a}{2} \end{matrix} \middle| \frac{3}{4} \right] \\ = \begin{cases} \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ -\frac{1}{3}+a, & \frac{1}{3}+a \end{matrix} \right]_m, & n=3m; \\ \frac{1}{1-3a} \left[ \begin{matrix} \frac{2}{3}, & \frac{4}{3} \\ \frac{1}{3}+a, & \frac{2}{3}+a \end{matrix} \right]_m, & n=1+3m; \\ 0, & n=2+3m. \end{cases} \end{aligned}$$

**Example 8** ( $\ell = 2$  in Theorem 4).

$$\begin{aligned} {}_3F_2 \left[ \begin{matrix} -n, & a, & 3a+n-2 \\ \frac{3a-3}{2}, & \frac{3a}{2} \end{matrix} \middle| \frac{3}{4} \right] \\ = \begin{cases} \frac{(3a-2)(a+2m-1)}{(a-1)(3a+6m-2)} \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ -\frac{1}{3}+a, & -\frac{2}{3}+a \end{matrix} \right]_m, & n=3m; \\ \frac{2}{3(1-a)} \left[ \begin{matrix} \frac{2}{3}, & \frac{4}{3} \\ -\frac{1}{3}+a, & \frac{1}{3}+a \end{matrix} \right]_m, & n=1+3m; \\ \frac{2(a+2m+1)}{(a-1)(3a-1)(3a+6m+2)} \left[ \begin{matrix} \frac{4}{3}, & \frac{5}{3} \\ \frac{1}{3}+a, & \frac{2}{3}+a \end{matrix} \right]_m, & n=2+3m. \end{cases} \end{aligned}$$

### 3. SEVERAL IDENTITIES FROM REVERSAL

Performing the replacement  $k \rightarrow n-k$  and  $a \rightarrow q^{1-2n}/b$  in Theorem 1, we derive the following equation.

**Theorem 5.** For a nonnegative integer  $\ell$  and a complex number  $b$ , there holds

$$\begin{aligned} \sum_{k=0}^n \frac{1}{(q^{2-n}b; q^3)_k} \left[ \begin{matrix} q^{-n}, \sqrt{b}, -\sqrt{b}, \sqrt{qb}, -\sqrt{qb} \\ q, q^{\ell}b \end{matrix} \middle| q \right]_k q^{(1+\ell)k} &= \frac{(q^{1-2n}/b; q)_n}{(q^{1-2n}/b; q^3)_n} \\ &\times \sum_{i=0}^{\ell} (-1)^n \frac{q^{\{(4+4n+6\ell-6i)i-n-n^2\}/6}}{b^{(n+2i)/3}} \frac{q-q^{2i}b}{q-b} \left[ \begin{matrix} q^{-\ell}, q^{-n}, b/q \\ q, q^{\ell}b, q^{1-n-i}/b \end{matrix} \middle| q \right]_i \\ &\times \left[ \begin{matrix} q, q^2 \\ q^{2-2n}/b, q^{3-2n}/b \end{matrix} \middle| q^3 \right]_{\frac{n-i}{3}} \chi(n-i). \end{aligned}$$

When  $\ell = 0$ , Theorem 5 reduces to (5) exactly. Other two examples are displayed as follows.

**Example 9** ( $\ell = 1$  in Theorem 5).

$$\begin{aligned} & \sum_{k=0}^n \frac{1}{(q^{2-n}b; q^3)_k} \left[ \begin{matrix} q^{-n}, \sqrt{b}, -\sqrt{b}, \sqrt{qb}, -\sqrt{qb} \\ q, qb \end{matrix} \middle| q \right]_k q^{2k} \\ &= \begin{cases} \left[ \begin{matrix} q, q^2 \\ q/b, q^2b \end{matrix} \middle| q^3 \right]_m, & n = 3m; \\ \frac{1-q}{1-qb} \left[ \begin{matrix} q^2, q^4 \\ q^2/b, q^4b \end{matrix} \middle| q^3 \right]_m, & n = 1 + 3m; \\ 0, & n = 2 + 3m. \end{cases} \end{aligned}$$

**Example 10** ( $\ell = 2$  in Theorem 5).

$$\begin{aligned} & \sum_{k=0}^n \frac{1}{(q^{2-n}b; q^3)_k} \left[ \begin{matrix} q^{-n}, \sqrt{b}, -\sqrt{b}, \sqrt{qb}, -\sqrt{qb} \\ q, q^2b \end{matrix} \middle| q \right]_k q^{3k} \\ &= \begin{cases} \left[ \begin{matrix} q, q^2 \\ q/b, q^2b \end{matrix} \middle| q^3 \right]_m, & n = 3m; \\ \frac{1-q^2}{1-q^2b} \left[ \begin{matrix} q^2, q^4 \\ q^2/b, q^4b \end{matrix} \middle| q^3 \right]_m, & n = 1 + 3m; \\ \frac{(1-q)(1-q^2)}{(1-q^2b)(1-q^3b)} \left[ \begin{matrix} q^4, q^5 \\ q^3/b, q^6b \end{matrix} \middle| q^3 \right]_m, & n = 2 + 3m. \end{cases} \end{aligned}$$

Employing the substitution  $a \rightarrow q^{3a}$  in Theorem 5 and then letting  $q \rightarrow 1$ , we get the following equation.

**Theorem 6.** *For a nonnegative integer  $\ell$  and a complex number  $a$ , there holds*

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} -n, & \frac{3b}{2}, & \frac{1+3b}{2} \\ 3b+\ell, & b-\frac{n-2}{3} \end{matrix} \middle| \frac{4}{3} \right] = \left[ \frac{3b+n}{3} - b \right]_n \\ & \times \sum_{i=0}^{\ell} \frac{(-1)^i}{3^n} \frac{3b-1+2i}{3b-1} \left[ \begin{matrix} -\ell, -n, 3b-1 \\ 1, 3b+n, 3b+\ell \end{matrix} \right]_i \\ & \times \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ \frac{2-2n-3b}{3}, & \frac{3-2n-3b}{3} \end{matrix} \right]_{\frac{n-i}{3}} \chi(n-i). \end{aligned}$$

When  $\ell = 0$ , Theorem 5 reduces to (2) exactly. Other two examples are laid out as follows.

**Example 11** ( $\ell = 1$  in Theorem 5: Chen and Chu [5, Theorem 4]).

$$\begin{aligned} & {}_3F_2 \left[ \begin{matrix} -n, & \frac{3b}{2}, & \frac{1+3b}{2} \\ 1+3b, & b-\frac{n-2}{3} \end{matrix} \middle| \frac{4}{3} \right] \\ &= \begin{cases} \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ \frac{1}{3}-b, & \frac{2}{3}+b \end{matrix} \right]_m, & n = 3m; \\ \frac{1}{3b+1} \left[ \begin{matrix} \frac{2}{3}, & \frac{4}{3} \\ \frac{2}{3}-b, & \frac{4}{3}+b \end{matrix} \right]_m, & n = 1 + 3m; \\ 0, & n = 2 + 3m. \end{cases} \end{aligned}$$

**Example 12** ( $\ell = 2$  in Theorem 5: Chen and Chu [5, Theorem 25]).

$$\begin{aligned}
& {}_3F_2 \left[ \begin{matrix} -n, & \frac{3b}{2}, & \frac{1+3b}{2} \\ & 2+3b, & b - \frac{n-2}{3} \end{matrix} \middle| \frac{4}{3} \right] \\
&= \begin{cases} \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ \frac{1}{3}-b, & \frac{2}{3}+b \end{matrix} \middle| q \right]_m, & n = 3m; \\ \frac{2}{3b+2} \left[ \begin{matrix} \frac{2}{3}, & \frac{4}{3} \\ \frac{2}{3}-b, & \frac{4}{3}+b \end{matrix} \middle| q \right]_m, & n = 1+3m; \\ \frac{2}{3(b+1)(3b+2)} \left[ \begin{matrix} \frac{4}{3}, & \frac{5}{3} \\ 1-b, & 2+b \end{matrix} \middle| q \right]_m, & n = 2+3m. \end{cases}
\end{aligned}$$

Performing the replacement  $k \rightarrow n - k$  and  $a \rightarrow q^{1-2n}/b$  in Theorem 3, we deduce the following equation.

**Theorem 7.** For a nonnegative integer  $\ell$  and a complex number  $b$ , there holds

$$\begin{aligned}
& \sum_{k=0}^n \frac{1}{(q^{2-n}b; q^3)_k} \left[ \begin{matrix} q^{-n}, q^\ell \sqrt{b}, -\sqrt{b}, \sqrt{qb}, -\sqrt{qb} \\ q, q^\ell b \end{matrix} \middle| q \right]_k q^k = \frac{(q^{1-2n}/b; q)_n}{(q^{1-2n}/b; q^3)_n} \\
& \times \sum_{i=0}^{\ell} (-1)^n \frac{q^{\{(10n+6\ell-2)i-n-n^2\}/6}}{b^{(2n-5i)/3}} \frac{q - q^{2i}b}{q - b} \left[ \begin{matrix} q^{-\ell}, q^{-n}, b/q \\ q, q^\ell b, q^n b \end{matrix} \middle| q \right]_i \\
& \times \left[ \begin{matrix} q, q^2 \\ q^{2-2n}/b, q^{3-2n}/b \end{matrix} \middle| q^3 \right]_{\frac{n-i}{3}} \chi(n-i).
\end{aligned}$$

When  $\ell = 0$ , Theorem 7 reduces to (5) exactly. Other two examples are displayed as follows.

**Example 13** ( $\ell = 1$  in Theorem 7).

$$\begin{aligned}
& \sum_{k=0}^n \frac{1}{(q^{2-n}b; q^3)_k} \left[ \begin{matrix} q^{-n}, q\sqrt{b}, -\sqrt{b}, \sqrt{qb}, -\sqrt{qb} \\ q, qb \end{matrix} \middle| q \right]_k q^k \\
&= \begin{cases} \left[ \begin{matrix} q, q^2 \\ q/b, q^2b \end{matrix} \middle| q^3 \right]_m, & n = 3m; \\ \frac{\sqrt{b}(q-1)}{1-qb} \left[ \begin{matrix} q^2, q^4 \\ q^2/b, q^4b \end{matrix} \middle| q^3 \right]_m, & n = 1+3m; \\ 0, & n = 2+3m. \end{cases}
\end{aligned}$$

**Example 14** ( $\ell = 2$  in Theorem 7).

$$\begin{aligned}
& \sum_{k=0}^n \frac{1}{(q^{2-n}b; q^3)_k} \left[ \begin{matrix} q^{-n}, q^2\sqrt{b}, -\sqrt{b}, \sqrt{qb}, -\sqrt{qb} \\ q, q^2b \end{matrix} \middle| q \right]_k q^k \\
&= \begin{cases} \left[ \begin{matrix} q, q^2 \\ q/b, q^2b \end{matrix} \middle| q^3 \right]_m, & n = 3m; \\ \frac{\sqrt{b}(q^2-1)}{1-q^2b} \left[ \begin{matrix} q^2, q^4 \\ q^2/b, q^4b \end{matrix} \middle| q^3 \right]_m, & n = 1+3m; \\ \frac{qb(1-q)(1-q^2)}{(1-q^2b)(1-q^3b)} \left[ \begin{matrix} q^4, q^5 \\ q^3/b, q^6b \end{matrix} \middle| q^3 \right]_m, & n = 2+3m. \end{cases}
\end{aligned}$$

Employing the substitution  $a \rightarrow q^{3a}$  in Theorem 7 and then letting  $q \rightarrow 1$ , we obtain the following equation.



**Theorem 8.** For a nonnegative integer  $\ell$  and a complex number  $a$ , there holds

$$\begin{aligned} {}_3F_2 \left[ \begin{matrix} -n, & \frac{3b}{2} + \ell, & \frac{1+3b}{2} \\ 3b + \ell, & b - \frac{n-2}{3} & \left| \frac{4}{3} \right. \end{matrix} \right] &= \frac{1}{3^n} \left[ \frac{3b+n}{\frac{1-2n}{3} - b} \right]_n \\ &\times \sum_{i=0}^{\ell} \frac{3b-1+2i}{3b-1} \left[ \begin{matrix} -\ell, -n, 3b-1 \\ 1, 3b+n, 3b+\ell \end{matrix} \right]_i \\ &\times \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ \frac{2-2n-3b}{3}, & \frac{3-2n-3b}{3} \end{matrix} \right]_{\frac{n-i}{3}} \chi(n-i). \end{aligned}$$

When  $\ell = 0$ , Theorem 8 reduces to (2) exactly. Other two examples are laid out as follows.

**Example 15** ( $\ell = 1$  in Theorem 8).

$$\begin{aligned} {}_3F_2 \left[ \begin{matrix} -n, & \frac{1+3b}{2}, & \frac{2+3b}{2} \\ 1+3b, & b - \frac{n-2}{3} & \left| \frac{4}{3} \right. \end{matrix} \right] \\ = \begin{cases} \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ \frac{1}{3} - b, & \frac{2}{3} + b \end{matrix} \right]_m, & n = 3m; \\ \frac{-1}{3b+1} \left[ \begin{matrix} \frac{2}{3}, & \frac{4}{3} \\ \frac{2}{3} - b, & \frac{4}{3} + b \end{matrix} \right]_m, & n = 1 + 3m; \\ 0, & n = 2 + 3m. \end{cases} \end{aligned}$$

**Example 16** ( $\ell = 2$  in Theorem 8).

$$\begin{aligned} {}_3F_2 \left[ \begin{matrix} -n, & \frac{1+3b}{2}, & \frac{4+3b}{2} \\ 2+3b, & b - \frac{n-2}{3} & \left| \frac{4}{3} \right. \end{matrix} \right] \\ = \begin{cases} \left[ \begin{matrix} \frac{1}{3}, & \frac{2}{3} \\ \frac{1}{3} - b, & \frac{2}{3} + b \end{matrix} \right]_m, & n = 3m; \\ \frac{-2}{3b+2} \left[ \begin{matrix} \frac{2}{3}, & \frac{4}{3} \\ \frac{2}{3} - b, & \frac{4}{3} + b \end{matrix} \right]_m, & n = 1 + 3m; \\ \frac{2}{3(b+1)(3b+2)} \left[ \begin{matrix} \frac{4}{3}, & \frac{5}{3} \\ 1-b, & 2+b \end{matrix} \right]_m, & n = 2 + 3m. \end{cases} \end{aligned}$$

Although Theorems 6 and 8 can produce countless  ${}_3F_2(\frac{4}{3})$ -series identities related to (2) with the change of  $\ell$ , many known results due to Chu and Chen [4, 5] can't be covered by them.

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